

Generalized twist deformations of Poincare and Galilei Hopf algebras

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Abstract

The three new deformed Poincare Hopf algebras are constructed with use of twist procedure. The corresponding relativistic space-times providing the sum of canonical and Lie-algebraic type of noncommutativity are proposed. Finally, the nonrelativistic contraction limits to the corresponding Galilei Hopf algebras are performed.

1 Introduction

Due to several theoretical arguments (see e.g. [1]-[4]) the interest in studying of space-time noncommutativity is growing rapidly. In accordance with the Hopf-algebraic classification of all deformations of relativistic and nonrelativistic symmetries (see [5], [6]) the most general form of space-time noncommutativity looks as follows

$$[x_\mu, x_\nu] = \theta_{\mu\nu}(x) , \quad (1)$$

where

$$\theta_{\mu\nu}(x) = \theta_{\mu\nu}^{(0)} + \theta_{\mu\nu}^{(1)\rho} x_\rho + \theta_{\mu\nu}^{(2)\rho\tau} x_\rho x_\tau . \quad (2)$$

For the simplest, canonical noncommutativity ($\theta_{\mu\nu}(x) = \theta_{\mu\nu}^{(0)}$), the corresponding Poincare Hopf algebra has been provided in [7] and [8] with the use of twist procedure [9]-[11], while its nonrelativistic counterparts have been discovered by various contraction schemes in [12].

The Lie-algebraic ($\theta_{\mu\nu}(x) = \theta_{\mu\nu}^{(1)\rho} x_\rho$) relativistic and nonrelativistic symmetries have been proposed in [13] and [14] respectively. In the literature they are known as κ -Poincare and κ -Galilei Hopf algebra, which in relativistic case correspond to the following κ -Minkowski space-times

$$[x_0, x_i] = \frac{i}{\kappa} x_i \quad , \quad [x_i, x_j] = 0 , \quad (3)$$

with mass-like deformation parameter κ .

Besides, there were proposed the twist deformations of a Lie-type at relativistic and nonrelativistic level in [15], [16] and [12].

The quadratic deformation ($\theta_{\mu\nu}(x) = \theta_{\mu\nu}^{(2)\rho\tau} x_\rho x_\tau$) has been studied in [17] and [15].

Unfortunately, in almost all theoretical considerations the mentioned above quantum space-times are considered separately. Recently, however, there was proposed in [18] (see also [19]) the relativistic Hopf structure corresponding to the so-called generalized space-time, with coefficients $\theta_{\mu\nu}^{(0)}$ and $\theta_{\mu\nu}^{(1)\rho}$ different than zero simultaneously. Particular, it has been shown that by canonical twist deformation of κ -Poincare Hopf algebra, we get $(\theta_{\mu\nu}, \kappa)$ -deformed symmetries associated with the following quantum space

$$[x_0, x_i] = \frac{i}{\kappa} x_i + i\theta_{0i} \quad , \quad [x_i, x_j] = i\theta_{ij} . \quad (4)$$

In this article we propose three new Poincare Hopf universal enveloping algebras $\mathcal{U}_{\theta_{kl}, \kappa}(\mathcal{P})$, $\mathcal{U}_{\theta_{0i}, \hat{\kappa}}(\mathcal{P})$ and $\mathcal{U}_{\theta_{0i}, \bar{\kappa}}(\mathcal{P})$ corresponding to the following generalized space-times $(a, b = 1, 2, 3)^1$

$$[x_0, x_a] = \frac{i}{\kappa} x_i \delta_{ak} \quad , \quad [x_a, x_b] = 2i\theta_{kl}(\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}) + \frac{i}{\kappa} x_0(\delta_{ia}\delta_{kb} - \delta_{ka}\delta_{ib}) , \quad (5)$$

¹The mentioned space-times are defined as the Hopf modules of $\mathcal{U}_{\theta_{kl}, \kappa}(\mathcal{P})$, $\mathcal{U}_{\theta_{0i}, \hat{\kappa}}(\mathcal{P})$ and $\mathcal{U}_{\theta_{0i}, \bar{\kappa}}(\mathcal{P})$ Hopf algebras respectively (see e.g. [20], [21], [8]).

$$[x_0, x_a] = \frac{i}{\hat{\kappa}}(\delta_{la}x_k - \delta_{ka}x_l) + 2i\theta_{0i}\delta_{ia} \quad , \quad [x_a, x_b] = 0 \quad , \quad (6)$$

and

$$[x_0, x_a] = 2i\theta_{0i}\delta_{ia} \quad , \quad [x_a, x_b] = \frac{i}{\hat{\kappa}}\delta_{ib}(\delta_{ka}x_l - \delta_{la}x_k) + \frac{i}{\hat{\kappa}}\delta_{ia}(\delta_{lb}x_k - \delta_{kb}x_l) \quad , \quad (7)$$

respectively. All three examples (5)-(7) are obtained by the twisting of classical algebra with use of the factors defined as suitable superposition of twist factors for canonical and Lie-algebraic deformation of relativistic symmetry. In other words, used in this article algorithm follows the procedure [9]-[11] used in [18], [19], but this time, besides twisting the κ -Poincare algebra, we supplement with second twist factor the twisted Poincare Hopf algebras [8], [15]. Further, in the second step of our investigation, we also perform three nonrelativistic contractions ([22], [23]; see also [12]) of our generalized Poincare Hopf structures. In such a way we get the corresponding Galilei Hopf universal enveloping algebras $\mathcal{U}_{\xi_{kl}, \lambda}(\mathcal{G})$, $\mathcal{U}_{\xi_{0i}, \hat{\lambda}}(\mathcal{G})$ and $\mathcal{U}_{\xi_{0i}, \bar{\lambda}}(\mathcal{G})$ respectively.

It should be noted that there are several motivations for present studies. First of all, such investigations are interesting because they provide six new explicit Hopf algebras. Besides, it should be noted, that the presented algebras permit to construct the corresponding phase-spaces (see e.g. [24], [25]) in the framework of so-called Heisenberg double procedure [11]. Consequently, it permits us to discuss of Heisenberg uncertainty principle associated with such generalized quantum space-times. Finally, one can consider corresponding classical and quantum relativistic and the nonrelativistic particle models. Such a construction has been already presented in the case of classical nonrelativistic particle moving in external constant force [26], [27], and the studies of deformations (5)-(7) in a context of dynamical considerations are postponed for further investigations.

The paper is organized as follows. In second Section we recall necessary facts concerning twist-deformed Poincare Hopf algebras [8], [15]. In Section 3 we present three new Poincare Hopf structures; the corresponding generalized space-times and the proper non-relativistic contractions to Galilei algebras are presented in Sections 4 and 5 respectively. The results are summarized and discussed in the last Section.

2 Twisted Poincare Hopf algebras

Let us recall five canonically and Lie-algebraically twisted Poincare Hopf algebras $\mathcal{U}(\mathcal{P})$ proposed in [8] and [15] respectively. All of them are described by classical (undeformed) algebraic sector ($\eta_{\mu\nu} = (-, +, +, +)$)

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\sigma}M_{\mu\rho} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma}) \quad , \\ [M_{\mu\nu}, P_\rho] &= i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu) \quad , \quad [P_\mu, P_\nu] = 0 \quad , \end{aligned} \quad (8)$$

and twisted coalgebraic part

$$\Delta_0(a) \rightarrow \Delta.(a) = \mathcal{F} \circ \Delta_0(a) \circ \mathcal{F}^{-1} \quad , \quad S.(a) = u.S_0(a)u^{-1} \quad , \quad (9)$$

with $\Delta_0(a) = a \otimes 1 + 1 \otimes a$, $S_0(a) = -a$ and $u. = \sum f_{(1)} S_0(f_{(2)})$ (we use Sweedler's notation $\mathcal{F} = \sum f_{(1)} \otimes f_{(2)}$).

Present in the above formula twist element $\mathcal{F} \in \mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P})$ satisfies the classical cocycle condition [10], [11]

$$\mathcal{F}_{.12} \cdot (\Delta_0 \otimes 1) \mathcal{F} = \mathcal{F}_{.23} \cdot (1 \otimes \Delta_0) \mathcal{F} , \quad (10)$$

and the normalization condition

$$(\epsilon \otimes 1) \mathcal{F} = (1 \otimes \epsilon) \mathcal{F} = 1 , \quad (11)$$

with $\mathcal{F}_{.12} = \mathcal{F} \otimes 1$ and $\mathcal{F}_{.23} = 1 \otimes \mathcal{F}$.

In the case of first, canonically deformed algebra $\mathcal{U}_{\theta_{kl}}(\mathcal{P})$, the twist element looks as follows ($\theta_{kl} = -\theta_{lk}$)

$$\mathcal{F}_{\theta_{kl}} = \exp i (\theta_{kl} P_k \wedge P_l) \quad ; \quad [k, l - \text{fixed}] \quad , \quad a \wedge b = a \otimes b - b \otimes a \quad (12)$$

and, in accordance with (9), the corresponding coproduct sector takes the form

$$\Delta_{\theta_{kl}}(P_\mu) = \Delta_0(P_\mu) , \quad (13)$$

$$\begin{aligned} \Delta_{\theta_{kl}}(M_{\mu\nu}) &= \Delta_0(M_{\mu\nu}) - \theta_{kl} [(\eta_{k\mu} P_\nu - \eta_{k\nu} P_\mu) \otimes P_l + P_k \otimes (\eta_{l\mu} P_\nu - \eta_{l\nu} P_\mu)] \\ &+ \theta_{kl} [(\eta_{l\mu} P_\nu - \eta_{l\nu} P_\mu) \otimes P_k + P_l \otimes (\eta_{k\mu} P_\nu - \eta_{k\nu} P_\mu)] . \end{aligned} \quad (14)$$

The antipodes and counits remain undeformed

$$S_0(P_\mu) = -P_\mu \quad , \quad S_0(M_{\mu\nu}) = -M_{\mu\nu} \quad , \quad \epsilon(M_{\mu\nu}) = \epsilon(P_\mu) = 0 . \quad (15)$$

For the second considered algebra $\mathcal{U}_{\theta_{0i}}(\mathcal{P})$ we have the following twist element

$$\mathcal{F}_{\theta_{0i}} = \exp i (\theta_{0i} P_0 \wedge P_i) \quad ; \quad [i - \text{fixed}] \quad , \quad (\theta_{0i} = -\theta_{i0}) , \quad (16)$$

and the corresponding coproduct sector is given by

$$\Delta_{\theta_{0i}}(P_\mu) = \Delta_0(P_\mu) , \quad (17)$$

$$\begin{aligned} \Delta_{\theta_{0i}}(M_{\mu\nu}) &= \Delta_0(M_{\mu\nu}) - \theta_{0i} [(\eta_{0\mu} P_\nu - \eta_{0\nu} P_\mu) \otimes P_i + P_0 \otimes (\eta_{i\mu} P_\nu - \eta_{i\nu} P_\mu)] \\ &+ \theta_{0i} [(\eta_{i\mu} P_\nu - \eta_{i\nu} P_\mu) \otimes P_0 + P_i \otimes (\eta_{0\mu} P_\nu - \eta_{0\nu} P_\mu)] . \end{aligned} \quad (18)$$

The antipodes and counits become classical.

In the case of Lie-algebraically deformed Hopf algebra $\mathcal{U}_\kappa(\mathcal{P})$ the twist factor and coproducts look as follows

$$\mathcal{F}_\kappa = \exp \frac{i}{2\kappa} (P_k \wedge M_{i0}) \quad [i, k - \text{fixed}, \quad i \neq k] , \quad (19)$$

²All carriers of considered twist factors are Abelian.

$$\begin{aligned}
\Delta_\kappa(P_\mu) &= \Delta_0(P_\mu) + \sinh\left(\frac{1}{2\kappa}P_k\right) \wedge (\eta_{i\mu}P_0 - \eta_{0\mu}P_i) \\
&+ \left(\cosh\left(\frac{1}{2\kappa}P_k\right) - 1\right) \perp (\eta_{i\mu}P_i - \eta_{0\mu}P_0) ,
\end{aligned} \tag{20}$$

$$\begin{aligned}
\Delta_\kappa(M_{\mu\nu}) &= \Delta_0(M_{\mu\nu}) + \frac{1}{2\kappa}M_{i0} \wedge (\eta_{\mu k}P_\nu - \eta_{\nu k}P_\mu) \\
&+ i[M_{\mu\nu}, M_{i0}] \wedge \sinh\left(\frac{1}{2\kappa}P_k\right) \\
&- [[M_{\mu\nu}, M_{i0}], M_{i0}] \perp \left(\cosh\left(\frac{1}{2\kappa}P_k\right) - 1\right) \\
&+ \frac{1}{2\kappa}M_{i0} \sinh\left(\frac{1}{2\kappa}P_k\right) \perp (\psi_k P_i - \chi_k P_0) \\
&- \frac{1}{2\kappa}(\psi_k P_0 - \chi_k P_i) \wedge M_{i0} \left(\cosh\left(\frac{1}{2\kappa}P_k\right) - 1\right) ,
\end{aligned} \tag{21}$$

where

$$\psi_k = \delta_{\nu k}\delta_{0\mu} - \delta_{\mu k}\delta_{0\nu} \quad , \quad \chi_k = \delta_{\nu k}\delta_{i\mu} - \delta_{\mu k}\delta_{i\nu} \quad , \quad a \perp b = a \otimes b + b \otimes a . \tag{22}$$

The antipodes and coproducts remain undeformed.

In the case of two remaining Lie-algebraic Hopf structures the twist factors are given by

$$\mathcal{F}_{\hat{\kappa}} = \exp \frac{i}{2\hat{\kappa}} (P_0 \wedge M_{kl}) \quad [k, l - \text{fixed}] , \tag{23}$$

$$\mathcal{F}_{\bar{\kappa}} = \exp \frac{i}{2\bar{\kappa}} (P_i \wedge M_{kl}) \quad [i, k, l - \text{fixed}, \quad i \neq k, l] , \tag{24}$$

while the corresponding coproducts take the form

$$\begin{aligned}
\Delta_{\hat{\kappa}}(P_\mu) &= \Delta_0(P_\mu) + \sin\left(\frac{1}{2\hat{\kappa}}P_0\right) \wedge (\eta_{k\mu}P_l - \eta_{l\mu}P_k) \\
&+ \left(\cos\left(\frac{1}{2\hat{\kappa}}P_0\right) - 1\right) \perp (\eta_{k\mu}P_k + \eta_{l\mu}P_l) ,
\end{aligned} \tag{25}$$

$$\begin{aligned}
\Delta_{\hat{\kappa}}(M_{\mu\nu}) &= \Delta_0(M_{\mu\nu}) + \frac{1}{2\hat{\kappa}}M_{kl} \wedge (\eta_{\mu 0}P_\nu - \eta_{\nu 0}P_\mu) \\
&+ i[M_{\mu\nu}, M_{kl}] \wedge \sin\left(\frac{1}{2\hat{\kappa}}P_0\right) \\
&+ [[M_{\mu\nu}, M_{kl}], M_{kl}] \perp \left(\cos\left(\frac{1}{2\hat{\kappa}}P_0\right) - 1\right) \\
&+ \frac{1}{2\hat{\kappa}}M_{kl} \sin\left(\frac{1}{2\hat{\kappa}}P_0\right) \perp (\psi_0 P_k - \chi_0 P_l) \\
&+ \frac{1}{2\hat{\kappa}}(\psi_0 P_l + \chi_0 P_k) \wedge M_{kl} \left(\cos\left(\frac{1}{2\hat{\kappa}}P_0\right) - 1\right) ,
\end{aligned} \tag{26}$$

and

$$\begin{aligned}\Delta_{\bar{\kappa}}(P_\mu) &= \Delta_0(P_\mu) + \sin\left(\frac{1}{2\bar{\kappa}}P_i\right) \wedge (\eta_{k\mu}P_l - \eta_{l\mu}P_k) \\ &+ \left(\cos\left(\frac{1}{2\bar{\kappa}}P_i\right) - 1\right) \perp (\eta_{k\mu}P_k + \eta_{l\mu}P_l) ,\end{aligned}\quad (27)$$

$$\begin{aligned}\Delta_{\bar{\kappa}}(M_{\mu\nu}) &= \Delta_0(M_{\mu\nu}) + \frac{1}{2\bar{\kappa}}M_{kl} \wedge (\eta_{\mu i}P_\nu - \eta_{\nu i}P_\mu) \\ &+ i[M_{\mu\nu}, M_{kl}] \wedge \sin\left(\frac{1}{2\bar{\kappa}}P_i\right) \\ &+ [[M_{\mu\nu}, M_{kl}], M_{kl}] \perp \left(\cos\left(\frac{1}{2\bar{\kappa}}P_i\right) - 1\right) \\ &+ \frac{1}{2\bar{\kappa}}M_{kl} \sin\left(\frac{1}{2\bar{\kappa}}P_i\right) \perp (\psi_i P_k - \chi_i P_l) \\ &+ \frac{1}{2\bar{\kappa}}(\psi_i P_l + \chi_i P_k) \wedge M_{kl} \left(\cos\left(\frac{1}{2\bar{\kappa}}P_i\right) - 1\right) ,\end{aligned}\quad (28)$$

respectively, with

$$\psi_\lambda = \eta_{\nu\lambda}\eta_{l\mu} - \eta_{\mu\lambda}\eta_{l\nu} \quad , \quad \chi_\lambda = \eta_{\nu\lambda}\eta_{k\mu} - \eta_{\mu\lambda}\eta_{k\nu} . \quad (29)$$

The antipodes and counit remain classical.

It should be noted, that all above algebras can be derived from relativistic classical r-matrices $r. \in \mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P})$, which are given by

$$r_{\theta_{kl}} = \theta_{kl}P_k \wedge P_l \quad [k, l - \text{fixed}] , \quad (30)$$

$$r_{\theta_{0i}} = \theta_{0i}P_0 \wedge P_i \quad [i - \text{fixed}] , \quad (31)$$

$$r_\kappa = \frac{1}{2\kappa}P_k \wedge M_{i0} \quad [i, k - \text{fixed}, i \neq k] , \quad (32)$$

$$r_{\hat{\kappa}} = \frac{1}{2\hat{\kappa}}P_0 \wedge M_{kl} \quad [k, l - \text{fixed}] , \quad (33)$$

and

$$r_{\bar{\kappa}} = \frac{1}{2\bar{\kappa}}P_i \wedge M_{kl} \quad [i, k, l - \text{fixed}, i \neq k, l] . \quad (34)$$

The matrices (30)-(34) satisfy the classical Yang-Baxter equation (CYBE)

$$[[r., r.]] = [r_{.12}, r_{.13} + r_{.23}] + [r_{.13}, r_{.23}] = 0 , \quad (35)$$

where the symbol $[[\cdot, \cdot]]$ denotes the Schouten bracket and $r_{.12} = r. \wedge 1$, $r_{.13} = r_{.1} \wedge 1 \wedge r_{.2}$, $r_{.23} = 1 \wedge r.$, $r. = r_{.1} \wedge r_{.2}$.

Obviously, for parameters θ_{kl} , θ_{0i} running to zero and parameters κ , $\hat{\kappa}$, $\bar{\kappa}$ approaching infinity all above algebras become classical.

3 Generalized twisted Poincare Hopf algebras

In this Section we introduced the generalized twisted algebras described by proper sums of matrices (30)-(34).

i) relativistic (θ_{kl}, κ) -deformation

Let us start with the following classical r-matrix

$$r_{\theta_{kl}, \kappa} = \frac{1}{2\kappa} P_k \wedge M_{i0} + \theta_{kl} P_k \wedge P_l, \quad (36)$$

defined as the sum of r-matrices (30) and (32) with indices k, l different than i ³. One can check that it satisfies the CYBE equation (35).

We find the corresponding deformed coproduct sector in two steps. Firstly, we twist classical Poincare algebra $\mathcal{U}_0(\mathcal{P})$ with use of the factor (12) or (19). In such a way we get the Hopf algebra $\mathcal{U}_{\theta_{kl}}(\mathcal{P})$ or $\mathcal{U}_{\kappa}(\mathcal{P})$ described in pervious Section. Next, following [18] and [19], we twist the coalgebraic sector of $\mathcal{U}_{\theta_{kl}}(\mathcal{P})$ or $\mathcal{U}_{\kappa}(\mathcal{P})$ with the use of twist factor⁴ satisfying θ_{kl} - or κ -deformed cocycle condition (see formulas (13), (14) or (20), (21))

$$\bar{\mathcal{F}}_{\kappa/\theta_{kl}12} \cdot (\Delta_{\theta_{kl}/\kappa} \otimes 1) \bar{\mathcal{F}}_{\kappa/\theta_{kl}} = \bar{\mathcal{F}}_{\kappa/\theta_{kl}23} \cdot (1 \otimes \Delta_{\theta_{kl}/\kappa}) \bar{\mathcal{F}}_{\kappa/\theta_{kl}}, \quad (37)$$

and normalization condition

$$(\epsilon \otimes 1) \bar{\mathcal{F}}_{\kappa/\theta_{kl}} = (1 \otimes \epsilon) \bar{\mathcal{F}}_{\kappa/\theta_{kl}} = 1. \quad (38)$$

One can check that the solutions $\bar{\mathcal{F}}_{\kappa/\theta_{kl}}$ of the equations (37), (38) are the same as for classical coproduct $\Delta_0(a)$, i.e. they are given by the formulas (19) or (12) respectively⁵. In such a way we get the following coalgebraic sector

$$\begin{aligned} \Delta_{\theta_{kl}, \kappa}(P_{\mu}) &= \Delta_0(P_{\mu}) + \sinh\left(\frac{1}{2\kappa} P_k\right) \wedge (\eta_{i\mu} P_0 - \eta_{0\mu} P_i) \\ &+ \left(\cosh\left(\frac{1}{2\kappa} P_k\right) - 1\right) \perp (\eta_{i\mu} P_i - \eta_{0\mu} P_0), \end{aligned} \quad (39)$$

$$\begin{aligned} \Delta_{\theta_{kl}, \kappa}(M_{\mu\nu}) &= \Delta_0(M_{\mu\nu}) + \frac{1}{2\kappa} M_{i0} \wedge (\eta_{\mu k} P_{\nu} - \eta_{\nu k} P_{\mu}) \\ &+ i [M_{\mu\nu}, M_{i0}] \wedge \sinh\left(\frac{1}{2\kappa} P_k\right) \\ &- [[M_{\mu\nu}, M_{i0}], M_{i0}] \perp \left(\cosh\left(\frac{1}{2\kappa} P_k\right) - 1\right) \\ &+ \frac{1}{2\kappa} M_{i0} \sinh\left(\frac{1}{2\kappa} P_k\right) \perp (\psi_k P_i - \chi_k P_0) \\ &- \frac{1}{2\kappa} (\psi_k P_0 - \chi_k P_i) \wedge M_{i0} \left(\cosh\left(\frac{1}{2\kappa} P_k\right) - 1\right) \end{aligned} \quad (40)$$

³The carrier of matrix (36) is Abelian.

⁴The considered (second) factor corresponds to the matrix (32) or (30) respectively.

⁵Consequently, in accordance with the Abelian character of matrix (36), the total twist factor takes the form $\mathcal{F}_{\theta_{kl}, \kappa} = \mathcal{F}_{\kappa} \cdot \mathcal{F}_{\theta_{kl}} = \mathcal{F}_{\theta_{kl}} \cdot \mathcal{F}_{\kappa} = \exp i(\frac{1}{2\kappa} P_k \wedge M_{i0} + \theta_{kl} P_k \wedge P_l)$.

$$\begin{aligned}
& - \theta^{kl}[(\eta_{k\mu}P_\nu - \eta_{k\nu}P_\mu) \otimes P_l + P_k \otimes (\eta_{l\mu}P_\nu - \eta_{l\nu}P_\mu)] \\
& + \theta_{kl}[(\eta_{l\mu}P_\nu - \eta_{l\nu}P_\mu) \otimes P_k + P_l \otimes (\eta_{k\mu}P_\nu - \eta_{k\nu}P_\mu)] \\
& + \theta_{kl}[[M_{\mu\nu}, M_{i0}], P_k] \perp \sinh(\frac{1}{2\kappa}P_k)P_l \\
& - \theta_{kl}[[M_{\mu\nu}, M_{i0}], P_l] \perp \sinh(\frac{1}{2\kappa}P_k)P_k \\
& + i\theta_{kl}[[[M_{\mu\nu}, M_{i0}], M_{i0}], P_k] \wedge (\cosh(\frac{1}{2\kappa}P_k) - 1)P_l \\
& - i\theta_{kl}[[[M_{\mu\nu}, M_{i0}], M_{i0}], P_l] \wedge (\cosh(\frac{1}{2\kappa}P_k) - 1)P_k,
\end{aligned}$$

which together with algebraic relations (8) and antipodes/counits (15) defines the generalized Poincare Hopf algebra $\mathcal{U}_{\theta_{kl}, \kappa}(\mathcal{P})$. It should be noted that for parameter θ_{kl} approaching zero we get $\mathcal{U}_\kappa(\mathcal{P})$ algebra, while for κ running to infinity we obtain $\mathcal{U}_{\theta_{kl}}(\mathcal{P})$ Poincare Hopf structure provided in pervious Section.

ii) relativistic $(\theta_{0i}, \hat{\kappa})$ -deformation

Let us now turn to the second (generalized) classical r-matrix

$$r_{\theta_{0i}, \hat{\kappa}} = \frac{1}{2\hat{\kappa}}P_0 \wedge M_{kl} + \theta_{0i}P_0 \wedge P_i, \quad (41)$$

where index i is different than k, l , ⁶. We see, that the above r-matrix is defined as a sum of r-matrices for $\mathcal{U}_{\hat{\kappa}}(\mathcal{P})$ and $\mathcal{U}_{\theta_{0i}}(\mathcal{P})$ algebras. Obviously, it satisfies CYBE equation (35).

In order to find the corresponding Hopf algebra we use the same prescription as in the case of constructed above Hopf structure $\mathcal{U}_{\theta_{kl}, \kappa}(\mathcal{P})$. It should be noted however, that this time, in the second step of used algorithm we twist the algebras $\mathcal{U}_{\hat{\kappa}}(\mathcal{P})$ or $\mathcal{U}_{\theta_{0i}}(\mathcal{P})$ respectively. One can check that as before, the corresponding twist factors are the same as (16) or (23), and they solve the modified cocycle condition with respect to coproducts (25), (26) or (17), (18)⁷. Consequently, we get

$$\begin{aligned}
\Delta_{\theta_{0i}, \hat{\kappa}}(P_\mu) &= \Delta_0(P_\mu) + \sin(\frac{1}{2\hat{\kappa}}P_0) \wedge (\eta_{k\mu}P_l - \eta_{l\mu}P_k) \\
&+ (\cos(\frac{1}{2\hat{\kappa}}P_0) - 1) \perp (\eta_{k\mu}P_k + \eta_{l\mu}P_l),
\end{aligned} \quad (42)$$

⁶The carrier of matrix (41) remains Abelian.

⁷Hence, the total twist factor can be written as $\mathcal{F}_{\theta_{0i}, \hat{\kappa}} = \exp i(\frac{1}{2\hat{\kappa}}P_0 \wedge M_{kl} + \theta_{0i}P_0 \wedge P_i)$.

$$\begin{aligned}
\Delta_{\theta_{0i}, \hat{\kappa}}(M_{\mu\nu}) &= \Delta_0(M_{\mu\nu}) + \frac{1}{2\hat{\kappa}} M_{kl} \wedge (\eta_{\mu 0} P_\nu - \eta_{\nu 0} P_\mu) \\
&+ i [M_{\mu\nu}, M_{kl}] \wedge \sin\left(\frac{1}{2\hat{\kappa}} P_0\right) \\
&+ [[M_{\mu\nu}, M_{kl}], M_{kl}] \perp (\cos\left(\frac{1}{2\hat{\kappa}} P_0\right) - 1) \\
&+ \frac{1}{2\hat{\kappa}} M_{kl} \sin\left(\frac{1}{2\hat{\kappa}} P_0\right) \perp (\psi_0 P_k - \chi_0 P_l) \\
&+ \frac{1}{2\hat{\kappa}} (\psi_0 P_l + \chi_0 P_k) \wedge M_{kl} (\cos\left(\frac{1}{2\hat{\kappa}} P_0\right) - 1) \\
&- \theta_{0i} [(\eta_{0\mu} P_\nu - \eta_{0\nu} P_\mu) \otimes P_i + P_0 \otimes (\eta_{i\mu} P_\nu - \eta_{i\nu} P_\mu)] \\
&+ \theta_{0i} [(\eta_{i\mu} P_\nu - \eta_{i\nu} P_\mu) \otimes P_0 + P_i \otimes (\eta_{0\mu} P_\nu - \eta_{0\nu} P_\mu)] \\
&+ \theta_{0i} [[M_{\mu\nu}, M_{kl}], P_0] \perp \sin\left(\frac{1}{2\hat{\kappa}} P_0\right) P_i \\
&- \theta_{0i} [[M_{\mu\nu}, M_{kl}], P_i] \perp \sin\left(\frac{1}{2\hat{\kappa}} P_0\right) P_0 \\
&- i\theta_{0i} [[[M_{\mu\nu}, M_{kl}], M_{kl}], P_0] \wedge (\cos\left(\frac{1}{2\hat{\kappa}} P_0\right) - 1) P_i \\
&+ i\theta_{0i} [[[M_{\mu\nu}, M_{kl}], M_{kl}], P_i] \wedge (\cos\left(\frac{1}{2\hat{\kappa}} P_0\right) - 1) P_0 .
\end{aligned} \tag{43}$$

The above relations together with algebraic sector (8) and antipodes/counits (15) define the generalized Poincare Hopf algebra $\mathcal{U}_{\theta_{0i}, \hat{\kappa}}(\mathcal{P})$. Of course, for parameter θ_{0i} running to zero we obtain Hopf structure $\mathcal{U}_{\hat{\kappa}}(\mathcal{P})$, while for parameter $\hat{\kappa}$ approaching infinity we get $\mathcal{U}_{\theta_{0i}}(\mathcal{P})$ algebra described in Section 2.

iii) relativistic $(\theta_{0i}, \bar{\kappa})$ -deformation

Let us now consider the last generalized r-matrix

$$r_{\theta_{0i}, \bar{\kappa}} = \frac{1}{2\bar{\kappa}} P_i \wedge M_{kl} + \theta_{0i} P_0 \wedge P_i , \tag{44}$$

defined as a sum of r-matrices for $\mathcal{U}_{\bar{\kappa}}(\mathcal{P})$ and $\mathcal{U}_{\theta_{0i}}(\mathcal{P})$ Hopf algebras with index i different than k, l and 0. It satisfies the classical Yang-Baxter equation (35).

We get the corresponding Hopf algebra by twist procedure of the coproducts (17), (18) or (27), (28), where the proper twist factors are exactly the same as (24) or (16), and satisfy the following modified cocycle condition

$$\bar{\mathcal{F}}_{\bar{\kappa}/\theta_{0i}12} \cdot (\Delta_{\theta_{0i}/\bar{\kappa}} \otimes 1) \bar{\mathcal{F}}_{\bar{\kappa}/\theta_{0i}} = \bar{\mathcal{F}}_{\bar{\kappa}/\theta_{0i}23} \cdot (1 \otimes \Delta_{\theta_{0i}/\bar{\kappa}}) \bar{\mathcal{F}}_{\bar{\kappa}/\theta_{0i}} , \tag{45}$$

respectively⁸. Then, we have

$$\begin{aligned}\Delta_{\theta_{0i},\bar{\kappa}}(P_\mu) &= \Delta_0(P_\mu) + \sin\left(\frac{1}{2\bar{\kappa}}P_i\right) \wedge (\eta_{k\mu}P_l - \eta_{l\mu}P_k) \\ &+ \left(\cos\left(\frac{1}{2\bar{\kappa}}P_i\right) - 1\right) \perp (\eta_{k\mu}P_k + \eta_{l\mu}P_l) ,\end{aligned}\tag{46}$$

$$\begin{aligned}\Delta_{\theta_{0i},\bar{\kappa}}(M_{\mu\nu}) &= \Delta_0(M_{\mu\nu}) + \frac{1}{2\bar{\kappa}}M_{kl} \wedge (\eta_{\mu i}P_\nu - \eta_{\nu i}P_\mu) \\ &+ i [M_{\mu\nu}, M_{kl}] \wedge \sin\left(\frac{1}{2\bar{\kappa}}P_i\right) \\ &+ [[M_{\mu\nu}, M_{kl}], M_{kl}] \perp \left(\cos\left(\frac{1}{2\bar{\kappa}}P_i\right) - 1\right) \\ &+ \frac{1}{2\bar{\kappa}}M_{kl} \sin\left(\frac{1}{2\bar{\kappa}}P_i\right) \perp (\psi_i P_k - \chi_i P_l) \\ &+ \frac{1}{2\bar{\kappa}}(\psi_i P_l + \chi_i P_k) \wedge M_{kl} \left(\cos\left(\frac{1}{2\bar{\kappa}}P_i\right) - 1\right) \\ &- \theta_{0i}[(\eta_{0\mu}P_\nu - \eta_{0\nu}P_\mu) \otimes P_i + P_0 \otimes (\eta_{i\mu}P_\nu - \eta_{i\nu}P_\mu)] \\ &+ \theta_{0i}[(\eta_{i\mu}P_\nu - \eta_{i\nu}P_\mu) \otimes P_0 + P_i \otimes (\eta_{0\mu}P_\nu - \eta_{0\nu}P_\mu)] \\ &+ \theta_{0i}[[M_{\mu\nu}, M_{kl}], P_0] \perp \sin\left(\frac{1}{2\bar{\kappa}}P_i\right)P_i \\ &- \theta_{0i}[[M_{\mu\nu}, M_{kl}], P_i] \perp \sin\left(\frac{1}{2\bar{\kappa}}P_i\right)P_0 \\ &- i\theta_{0i}[[[M_{\mu\nu}, M_{kl}], M_{kl}], P_0] \wedge \left(\cos\left(\frac{1}{2\bar{\kappa}}P_i\right) - 1\right)P_i \\ &+ i\theta_{0i}[[[M_{\mu\nu}, M_{kl}], M_{kl}], P_i] \wedge \left(\cos\left(\frac{1}{2\bar{\kappa}}P_i\right) - 1\right)P_0 .\end{aligned}\tag{47}$$

The above relations together with classical algebraic sector (8) and undeformed antipodes/counits (15) define generalized Poincare Hopf algebra $\mathcal{U}_{\theta_{0i},\bar{\kappa}}(\mathcal{P})$. Obviously, for $\bar{\kappa} \rightarrow \infty$ we get $\mathcal{U}_{\theta_{0i}}(\mathcal{P})$ Poincare Hopf algebra, while for $\theta_{0i} \rightarrow 0$ we obtain $\mathcal{U}_{\bar{\kappa}}(\mathcal{P})$ Hopf structure described in second Section.

4 Generalized twisted relativistic space-times

In this Section we introduce the generalized relativistic space-times corresponding to the Poincare Hopf algebras provided in pervious Section. They are defined as quantum representation spaces (Hopf modules) for quantum Poincare algebras, with action of the deformed symmetry generators satisfying suitably deformed Leibnitz rules [20], [21], [8]. The action of Poincare algebra on a Hopf module of functions depending on space-time coordinates x_μ is given by

⁸ $\mathcal{F}_{\theta_{0i},\bar{\kappa}} = \exp i\left(\frac{1}{2\bar{\kappa}}P_i \wedge M_{kl} + \theta_{0i}P_0 \wedge P_i\right).$

$$P_\mu \triangleright f(x) = i\partial_\mu f(x) \quad , \quad M_{\mu\nu} \triangleright f(x) = i(x_\mu \partial_\nu - x_\nu \partial_\mu) f(x) \quad , \quad (48)$$

while the \star -multiplication of arbitrary two functions is defined as follows

$$f(x) \star g(x) := \omega \circ (\mathcal{F}^{-1} \triangleright f(x) \otimes g(x)) \quad . \quad (49)$$

In the above formula \mathcal{F} denotes twist factor corresponding to a proper Poincare group and $\omega \circ (a \otimes b) = a \cdot b$.

i) the deformation (5)

Let us start with $\mathcal{U}_{\theta_{kl}, \kappa}(\mathcal{P})$ Hopf algebra described by matrix (36). In such a case, in accordance with considerations of pervious Section, the star multiplication of two arbitrary functions $f(x)$ and $g(x)$ is given by

$$f(x) \star_{\theta_{kl}, \kappa} g(x) = \omega \circ (\mathcal{F}_{\theta_{kl}, \kappa}^{-1} \triangleright (f(x) \otimes g(x))) \quad , \quad (50)$$

where the multiplication operator $\mathcal{F}_{\theta_{kl}, \kappa}$ is defined by the superposition of two twist factors (12) and (19)

$$\mathcal{F}_{\theta_{kl}, \kappa} = \mathcal{F}_{\theta_{kl}} \cdot \mathcal{F}_\kappa = \mathcal{F}_\kappa \cdot \mathcal{F}_{\theta_{kl}} = \exp \frac{i}{2} \left(2\theta_{kl}(P_k \wedge P_l) + \frac{1}{\kappa}(P_k \wedge M_{i0}) \right) \quad .$$

Hence, due to the formulas (48), (49) we have⁹

$$[x_0, x_a]_{\star_{\theta_{kl}, \kappa}} = \frac{i}{\kappa} x_i \delta_{ak} \quad , \quad [x_a, x_b]_{\star_{\theta_{kl}, \kappa}} = 2i\theta_{kl}(\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}) + \frac{i}{\kappa} x_0(\delta_{ia}\delta_{kb} - \delta_{ka}\delta_{ib}) \quad .$$

Of course, for parameter θ_{kl} approaching zero or parameter κ running to infinity, we get the relativistic space-times associated with Hopf algebras $\mathcal{U}_\kappa(\mathcal{P})$ or $\mathcal{U}_{\theta_{kl}}(\mathcal{P})$ respectively (see [8], [15]).

ii) the deformation (6)

In the case of $\mathcal{U}_{\theta_{0i}, \hat{\kappa}}(\mathcal{P})$ Hopf algebra the corresponding multiplication looks as follows

$$f(x) \star_{\theta_{0i}, \hat{\kappa}} g(x) = \omega \circ (\mathcal{F}_{\theta_{0i}, \hat{\kappa}}^{-1} \triangleright (f(x) \otimes g(x))) \quad , \quad (51)$$

with operator $\mathcal{F}_{\theta_{0i}, \hat{\kappa}}$ given by

$$\mathcal{F}_{\theta_{0i}, \hat{\kappa}} = \mathcal{F}_{\theta_{0i}} \cdot \mathcal{F}_{\hat{\kappa}} = \mathcal{F}_{\hat{\kappa}} \cdot \mathcal{F}_{\theta_{0i}} = \exp \frac{i}{2} \left(2\theta_{0i}(P_0 \wedge P_i) + \frac{1}{\hat{\kappa}}(P_0 \wedge M_{kl}) \right) \quad .$$

Consequently, we get

$$[x_0, x_a]_{\star_{\theta_{0i}, \hat{\kappa}}} = \frac{i}{\hat{\kappa}}(\delta_{ia}x_k - \delta_{ka}x_l) + 2i\theta_{0i}\delta_{ia} \quad , \quad [x_a, x_b]_{\star_{\theta_{0i}, \hat{\kappa}}} = 0 \quad . \quad (52)$$

The above relations define the relativistic space-time corresponding to the algebra $\mathcal{U}_{\theta_{0i}, \hat{\kappa}}(\mathcal{P})$. Obviously, for $\hat{\kappa} \rightarrow \infty$ or $\theta_{0i} \rightarrow 0$ we obtain the space-time associated with $\mathcal{U}_{\theta_{0i}}(\mathcal{P})$ or $\mathcal{U}_{\hat{\kappa}}(\mathcal{P})$ Hopf algebra provided in [8] and [15] respectively.

⁹ $[a, b]_\star = a \star b - b \star a$.

iii) the deformation (7)

For the last Hopf algebra $\mathcal{U}_{\theta_{0i}, \bar{\kappa}}(\mathcal{P})$ we have

$$f(x) \star_{\theta_{0i}, \bar{\kappa}} g(x) = \omega \circ (\mathcal{F}_{\theta_{0i}, \bar{\kappa}}^{-1} \triangleright (f(x) \otimes g(x))) , \quad (53)$$

and

$$\mathcal{F}_{\theta_{0i}, \bar{\kappa}} = \exp \frac{i}{2} \left(2\theta_{0i}(P_0 \wedge P_i) + \frac{1}{\bar{\kappa}}(P_i \wedge M_{kl}) \right) , \quad (54)$$

what gives the following relativistic space-time

$$[x_0, x_a]_{\star_{\theta_{0i}, \bar{\kappa}}} = 2i\theta_{0i}\delta_{ia} \quad , \quad [x_a, x_b]_{\star_{\theta_{0i}, \bar{\kappa}}} = \frac{i}{\bar{\kappa}}\delta_{ib}(\delta_{ka}x_l - \delta_{la}x_k) + \frac{i}{\bar{\kappa}}\delta_{ia}(\delta_{lb}x_k - \delta_{kb}x_l) .$$

Of course, for parameter $\bar{\kappa}$ approaching infinity or parameter θ_{0i} running to zero, we get the relativistic space-time corresponding to $\mathcal{U}_{\theta_{0i}}(\mathcal{P})$ or $\mathcal{U}_{\bar{\kappa}}(\mathcal{P})$ Poincare Hopf algebras respectively.

5 Generalized twisted Galilei Hopf algebras

In this Section we calculate the nonrelativistic contractions of Hopf structures derived in Section 3, i.e. we find their nonrelativistic counterparts - the generalized twist deformations of Galilei Hopf algebra.

First of all, let us introduce the following redefinition of Poincare generators [22]

$$P_0 = \frac{\Pi_0}{c} \quad , \quad P_i = \Pi_i \quad , \quad M_{ij} = K_{ij} \quad , \quad M_{i0} = cV_i , \quad (55)$$

where parameter c denotes the light velocity. Besides, we also introduce five parameters $\lambda, \hat{\lambda}, \bar{\lambda}, \xi_{kl} (\xi_{lk})$ and $\xi_{0i} (\xi_{i0})$ such that

$$\lambda = \kappa/c \quad , \quad \hat{\lambda} = \hat{\kappa}c \quad , \quad \bar{\lambda} = \bar{\kappa} \quad , \quad \xi_{kl} = \theta_{kl} \quad (\xi_{lk} = \theta_{lk}) \quad , \quad \xi_{0i} = \theta_{0i}/c \quad (\xi_{i0} = \theta_{i0}/c) . \quad (56)$$

Further, one performs the contraction limit of algebraic part (8) and coproducts ((39),(40)), ((42),(43)) and ((46),(47)) in two steps. Firstly, we rewrite the formulas (8) and ((39),(40)), ((42),(43)), ((46),(47)) in term of the operators (55) and parameters (56). Secondly, we take the $c \rightarrow \infty$ limit, and in such a way, we get the following algebraic¹⁰

$$\begin{aligned} [K_{ab}, K_{cd}] &= i(\delta_{ad}K_{bc} - \delta_{bd}K_{ac} + \delta_{bc}K_{ad} - \delta_{ac}K_{bd}) , \\ [K_{ab}, V_c] &= i(\delta_{bc}V_a - \delta_{ac}V_b) \quad , \quad [K_{ab}, \Pi_c] = i(\delta_{bc}\Pi_a - \delta_{ac}\Pi_b) , \\ [K_{ab}, \Pi_0] &= [V_a, V_b] = [V_a, \Pi_b] = 0 \quad , \quad [V_a, \Pi_0] = -i\Pi_a \quad , \quad [\Pi_\rho, \Pi_\sigma] = 0 , \end{aligned} \quad (57)$$

and coalgebraic sectors

¹⁰ $a, b, c, d = 1, 2, 3.$

i) nonrelativistic (ξ_{kl}, λ) -deformation

$$\Delta_{\xi_{kl}, \lambda}(\Pi_0) = \Delta_0(\Pi_0) + \frac{1}{2\lambda} \Pi_k \wedge \Pi_l, \quad (58)$$

$$\Delta_{\xi_{kl}, \lambda}(\Pi_a) = \Delta_0(\Pi_a), \quad \Delta_{\xi_{kl}, \lambda}(V_a) = \Delta_0(V_a), \quad (59)$$

$$\begin{aligned} \Delta_{\xi_{kl}, \lambda}(K_{ab}) &= \Delta_0(K_{ab}) + \frac{i}{2\lambda} [K_{ab}, V_i] \wedge \Pi_k + \frac{1}{2\lambda} V_i \wedge (\delta_{ak} \Pi_b - \delta_{bk} \Pi_a) \\ &\quad - \xi_{kl} [(\delta_{ka} \Pi_b - \delta_{kb} \Pi_a) \otimes \Pi_l + \Pi_k \otimes (\delta_{la} \Pi_b - \delta_{lb} \Pi_a)] \\ &\quad + \xi_{kl} [(\delta_{la} \Pi_b - \delta_{lb} \Pi_a) \otimes \Pi_k + \Pi_l \otimes (\delta_{ka} \Pi_b - \delta_{kb} \Pi_a)], \end{aligned} \quad (60)$$

ii) nonrelativistic $(\xi_{0i}, \hat{\lambda})$ -deformation

$$\begin{aligned} \Delta_{\xi_{0i}, \hat{\lambda}}(\Pi_\mu) &= \Delta_0(\Pi_\mu) + \sin\left(\frac{1}{2\hat{\lambda}} \Pi_0\right) \wedge (\delta_{k\mu} \Pi_l - \delta_{l\mu} \Pi_k) \\ &\quad + \left(\cos\left(\frac{1}{2\hat{\lambda}} \Pi_0\right) - 1\right) \perp (\delta_{k\mu} \Pi_k + \delta_{l\mu} \Pi_l), \end{aligned} \quad (61)$$

$$\begin{aligned} \Delta_{\xi_{0i}, \hat{\lambda}}(K_{ab}) &= \Delta_0(K_{ab}) + i [K_{ab}, K_{kl}] \wedge \sin\left(\frac{1}{2\hat{\lambda}} \Pi_0\right) \\ &\quad + [[K_{ab}, K_{kl}], K_{kl}] \perp \left(\cos\left(\frac{1}{2\hat{\lambda}} \Pi_0\right) - 1\right) \\ &\quad - \xi_{0i} \Pi_0 \wedge (\delta_{ia} \Pi_b - \delta_{ib} \Pi_a) \\ &\quad - \xi_{0i} [[K_{ab}, K_{kl}], \Pi_i] \perp \Pi_0 \sin\left(\frac{1}{2\hat{\lambda}} \Pi_0\right) \\ &\quad + i \xi_{0i} [[[K_{ab}, K_{kl}], K_{kl}], \Pi_i] \wedge \Pi_0 \left(\cos\left(\frac{1}{2\hat{\lambda}} \Pi_0\right) - 1\right), \end{aligned} \quad (62)$$

$$\begin{aligned} \Delta_{\xi_{0i}, \hat{\lambda}}(V_a) &= \Delta_0(V_a) + \frac{1}{2\hat{\lambda}} K_{kl} \wedge \Pi_a + i [V_a, K_{kl}] \wedge \sin\left(\frac{1}{2\hat{\lambda}} \Pi_0\right) \\ &\quad + [[V_a, K_{kl}], K_{kl}] \perp \left(\cos\left(\frac{1}{2\hat{\lambda}} \Pi_0\right) - 1\right) \\ &\quad + K_{kl} \sin\left(\frac{1}{2\hat{\lambda}} \Pi_0\right) \perp \frac{1}{2\hat{\lambda}} (\delta_{ka} \Pi_l - \delta_{la} \Pi_k) \\ &\quad - \frac{1}{2\hat{\lambda}} (\delta_{ka} \Pi_k + \delta_{la} \Pi_l) \wedge K_{kl} \left(\cos\left(\frac{1}{2\hat{\lambda}} \Pi_0\right) - 1\right) \\ &\quad - \xi_{0i} \Pi_a \wedge \Pi_i - i \xi_{0i} [[[V_a, K_{kl}], K_{kl}], \Pi_0] \wedge \left(\cos\left(\frac{1}{2\hat{\lambda}} \Pi_0\right) - 1\right) \Pi_i \\ &\quad + \xi_{0i} [[V_a, K_{kl}], \Pi_0] \perp \sin\left(\frac{1}{2\hat{\lambda}} \Pi_0\right) \Pi_i, \end{aligned} \quad (63)$$

iii) nonrelativistic $(\xi_{0i}, \bar{\lambda})$ -deformation

$$\begin{aligned}\Delta_{\xi_{0i}, \bar{\lambda}}(\Pi_\mu) &= \Delta_0(\Pi_\mu) + \sin\left(\frac{1}{2\bar{\lambda}}\Pi_i\right) \wedge (\delta_{k\mu}\Pi_l - \delta_{l\mu}\Pi_k) \\ &+ \left(\cos\left(\frac{1}{2\bar{\lambda}}\Pi_i\right) - 1\right) \perp (\delta_{k\mu}\Pi_k + \delta_{l\mu}\Pi_l) ,\end{aligned}\quad (64)$$

$$\begin{aligned}\Delta_{\xi_{0i}, \bar{\lambda}}(K_{ab}) &= \Delta_0(K_{ab}) + K_{kl} \wedge \frac{1}{2\bar{\lambda}} (\delta_{ai}\Pi_b - \delta_{bi}\Pi_a) \\ &+ i [K_{ab}, K_{kl}] \wedge \sin\left(\frac{1}{2\bar{\lambda}}\Pi_i\right) \\ &+ [[K_{ab}, K_{kl}], K_{kl}] \perp \left(\cos\left(\frac{1}{2\bar{\lambda}}\Pi_i\right) - 1\right) \\ &+ K_{kl} \sin\left(\frac{1}{2\bar{\lambda}}\Pi_i\right) \perp \frac{1}{2\bar{\lambda}} (\psi_i\Pi_k - \chi_i\Pi_l) \\ &+ \frac{1}{2\bar{\lambda}} (\psi_i\Pi_l + \chi_i\Pi_k) \wedge K_{kl} \left(\cos\left(\frac{1}{2\bar{\lambda}}\Pi_i\right) - 1\right)\end{aligned}\quad (65)$$

$$\begin{aligned}&- \xi_{0i}\Pi_0 \wedge (\delta_{ia}\Pi_b - \delta_{ib}\Pi_a) \\ &- \xi_{0i} [[K_{ab}, K_{kl}], \Pi_i] \perp \Pi_0 \sin\left(\frac{1}{2\bar{\lambda}}\Pi_i\right) \\ &+ i\xi_{0i} [[[K_{ab}, K_{kl}], K_{kl}], \Pi_i] \wedge \Pi_0 \left(\cos\left(\frac{1}{2\bar{\lambda}}\Pi_i\right) - 1\right) ,\end{aligned}$$

$$\begin{aligned}\Delta_{\xi_{0i}, \bar{\lambda}}(V_a) &= \Delta_0(V_a) + i [V_a, K_{kl}] \wedge \sin\left(\frac{1}{2\bar{\lambda}}\Pi_i\right) \\ &+ [[V_a, K_{kl}], K_{kl}] \perp \left(\cos\left(\frac{1}{2\bar{\lambda}}\Pi_i\right) - 1\right) \\ &- \xi_{0i}\Pi_a \wedge \Pi_i - i\xi_{0i} [[[V_a, K_{kl}], K_{kl}], \Pi_0] \wedge \left(\cos\left(\frac{1}{2\bar{\lambda}}\Pi_i\right) - 1\right)\Pi_i \\ &+ \xi_{0i} [[V_a, K_{kl}], \Pi_0] \perp \sin\left(\frac{1}{2\bar{\lambda}}\Pi_i\right)\Pi_i .\end{aligned}\quad (66)$$

The relations (57) together with coproducts **i**), **ii**) and **iii**) define the generalized twisted Galilei Hopf algebras $\mathcal{U}_{\xi_{kl}, \lambda}(\mathcal{G})$, $\mathcal{U}_{\xi_{0i}, \hat{\lambda}}(\mathcal{G})$ and $\mathcal{U}_{\xi_{0i}, \bar{\lambda}}(\mathcal{G})$ corresponding to the Poincare Hopf algebras $\mathcal{U}_{\theta_{kl}, \kappa}(\mathcal{P})$, $\mathcal{U}_{\theta_{0i}, \hat{\kappa}}(\mathcal{P})$ and $\mathcal{U}_{\theta_{0i}, \bar{\kappa}}(\mathcal{P})$ respectively. It should be noted that for parameters ξ_{kl} , ξ_{0i} running to zero and (or) parameters λ , $\hat{\lambda}$, $\bar{\lambda}$ approaching infinity, the above algebras become classical (or one gets twisted Galilei Hopf structures proposed in [12]).

6 Final remarks

In this article we provide three new generalized Poincare Hopf algebras and corresponding relativistic space-times. All three space-times combine two kinds of quantum deformations

- canonical and Lie-algebraic deformation leading to the models with quantum time and classical space as well as with quantum time and quantum space. Further, we also perform three nonrelativistic contraction limits to the corresponding Galilei Hopf structures.

The present studies can be extended in various ways. First of all, one can ask about $N = 1$ supersymmetric extensions of the constructed deformed Hopf algebras. Besides, one can also consider still more complicated (twisted) generalizations of the relativistic and nonrelativistic quantum space-times. For example, it is possible to consider the Poincare or Galilei Hopf structure leading to the superposition of canonically and quadratically deformed quantum space. Finally, it should be noted, that dual quantum Poincare (Galilei) groups $\mathcal{P}_{\theta_{kl}, \kappa}(\mathcal{G}_{\xi_{kl}, \lambda})$, $\mathcal{P}_{\theta_{0i}, \hat{\kappa}}(\mathcal{G}_{\xi_{0i}, \hat{\lambda}})$ and $\mathcal{P}_{\theta_{0i}, \bar{\kappa}}(\mathcal{G}_{\xi_{0i}, \bar{\lambda}})$ can be obtained by canonical quantization of the corresponding Poisson-Lie structure [28] or with use of so-called FRT procedure [29]. Consequently, as it was mentioned in Introduction, one can find in the framework of Heisenberg double procedure [11] the corresponding relativistic and nonrelativistic phase spaces associated with the above algebras. All these problems are now being studied.

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